

Ehrenfest theorems and charge antiscreening in Abelian projected gauge theories

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Abstract.

We derive exact relations for SU(2) lattice gauge theory in 3+1 dimensions. In terms of Abelian projection, these are the expectation values of Maxwell equations that define a new field strength operator and conserved, dynamic electric currents formed from the charged matter and ghost fields. The effect of gauge fixing is calculated, and in the maximally Abelian gauge we find antiscreening of U(1) Wilson loop source charges. We discuss the importance of these quantities in the dual superconducting vacuum mechanism of confinement.

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1 Introduction

Lattice studies based on Abelian projection have had considerable success identifying the dynamical variables relevant to the physics of quark confinement. There is no definitive way as yet of choosing the optimum variables, but in the maximally Abelian gauge [1, 2] the U(1) fields remaining after Abelian projection produce a heavy quark potential that continues to rise linearly [3]. Further the string tension is almost, but not exactly, equal to the full SU(2) quantity; 92% in a recent study at $\beta = 2.5115$ [4].

This suggests that we may be close to identifying an underlying principle governing confinement. All elements of a dual superconducting vacuum appear to be present [5, 1]; in the maximally Abelian gauge magnetic monopoles reproduce nearly all of the U(1) string tension [6, 4]. The spontaneous breaking to the U(1) gauge symmetry is signalled by the non-zero vacuum expectation value of monopole operator [7, 8]. The profile of the electric field and the persistent magnetic monopole currents in the vortex between quark and antiquark are well described by an effective theory, the Ginzburg–Landau, or equivalently a Higgs theory giving a London penetration depth and Ginzburg–Landau coherence length [9, 10].

Central to finding the effective theory is the definition of the field strength operator in the Abelian projected theory, entering not only in the vortex profiles but also in the formula for the monopole operator. All definitions should be equivalent in the continuum limit, but use of the appropriate lattice expression should lead to a minimisation of discretisation errors.

In this paper we exploit lattice symmetries to derive such an operator that satisfies Ehrenfest relations; Maxwell’s equations for ensemble averages irrespective of lattice artefacts. In section 2 we introduce and review this method in pure U(1) theories [11], discussing the Abelian projected SU(2) theory, with and without gauge fixing to the maximally Abelian gauge, in section 3.

The charged coset fields are normally discarded in Abelian projection, as are the ghost fields arising from the gauge fixing procedure. Since the remainder of the SU(2) infrared physics must arise from these, an understanding of their rôle is central to completing the picture of full SU(2) confinement. In section 3 we begin to address this issue, showing

that these fields form a charged $U(1)$ current, and in section 4 demonstrate that in the maximally Abelian gauge the supposedly unit charged Abelian Wilson loop has an upward renormalisation of charge due to this current of $O(15\%)$. A localised cloud of like polarity charge is induced in the vacuum in the vicinity of a source, producing an effect reminiscent of the antiscreening of charge in QCD. In other gauges studied, the analogous current is weaker, and acts to *screen* the source.

We show that this current can be quantitatively written as a sum of terms from the coset and ghost fields. The contribution of the ghost fields in the maximally Abelian gauge in this context is found to be small. The effect of the the Gribov ambiguity on these currents is argued to be slight.

Finally in section 5 we discuss these results. Some preliminary results have already appeared [12]. The reader's attention is drawn to related work in this subject [13, 14].

2 Abelian theories

The Wilson action

$$S_W = \sum_{n,\mu < \nu} (1 - \cos \theta_{\mu\nu}(n)) \quad (1)$$

comprises link angles $\{\theta_\mu(n) \in [-\pi, \pi)\}$ summed to form plaquette angles $\theta_{\mu\nu}(n) = \theta_\mu(n) + \theta_\nu(n + \hat{\mu}) - \theta_\mu(n + \hat{\nu}) - \theta_\nu(n)$. External electric sources may be represented by a Wilson loop, and we consider here for simplicity a specific plaquette on the lattice $P_{\kappa\lambda}(n') = \exp i\theta_{\kappa\lambda}(n')$ with real and imaginary parts $R_{\kappa\lambda}$, $I_{\kappa\lambda}$ respectively. The partition function

$$Z_{S\mathcal{RC}} = \int [d\theta_\mu] P_{\kappa\lambda}(n') e^{-\beta S_W}. \quad (2)$$

is invariant under the introduction of an arbitrary constant into any link angle. We call this a shift invariance and we focus on the consequence of this on a particular link: $\theta_\mu(n) \rightarrow \theta_\mu(n) + \varepsilon$. The shift corresponds to either a left or right multiplication of that link by a group element: $e^{i\theta_\mu(n)} \rightarrow e^{i\theta_\mu(n)} e^{i\varepsilon} = e^{i\varepsilon} e^{i\theta_\mu(n)}$. The Haar measure is invariant under this transformation: $d(\theta_\mu(n) + \varepsilon) = d\theta_\mu(n)$.

The first order shift in the partition function gives an identity:

$$\frac{1}{Z_{S\mathcal{RC}}} \frac{\partial Z_{S\mathcal{RC}}}{\partial \varepsilon} \Big|_{\varepsilon=0} = \frac{1}{Z_{S\mathcal{RC}}} \int [d\theta_\mu] \left[\frac{\partial P_{\kappa\lambda}(n')}{\partial \varepsilon} \Big|_{\varepsilon=0} - P_{\kappa\lambda}(n') \cdot \beta \frac{\partial S_W}{\partial \varepsilon} \Big|_{\varepsilon=0} \right] e^{-\beta S_W}$$

$$\begin{aligned}
&= \frac{1}{\int [d\theta_\mu] R_{\kappa\lambda} \cdot e^{-\beta S_W}} \int [d\theta_\mu] \left[i R_{\kappa\lambda} \cdot \delta_W - i I_{\kappa\lambda} \cdot \beta \frac{\partial S_W}{\partial \varepsilon} \Big|_{\varepsilon=0} \right] e^{-\beta S_W} \\
&= i \delta_W - \frac{i\beta}{\langle R_{\kappa\lambda} \rangle} \left\langle I_{\kappa\lambda} \cdot \frac{\partial S_W}{\partial \varepsilon} \Big|_{\varepsilon=0} \right\rangle = 0
\end{aligned} \tag{3}$$

where $\delta_W \neq 0$ only when the shifted link coincides with a link in the source,

$$\delta_W = \left[\delta_{\mu\kappa} (\delta_{n,n'} - \delta_{n,n'+\hat{\lambda}}) - \delta_{\mu\lambda} (\delta_{n,n'} - \delta_{n,n'+\hat{\kappa}}) \right]. \tag{4}$$

The real or imaginary part of $P_{\kappa\lambda}$ is dropped if it contributes a term odd in the link angle and hence has a zero expectation value.

Multiplying eqn. (4) by the electric charge e , where $\beta = 1/e^2$, we use the backwards lattice difference operator to define the lattice field strength tensor, $f_{\mu\nu}$

$$\frac{1}{e} \frac{\partial S_W}{\partial \varepsilon} \Big|_{\varepsilon=0} = \frac{1}{e} \Delta_\nu^- I_{\mu\nu}(n) \equiv \Delta_\nu^- f_{\mu\nu}(n). \tag{5}$$

The identity becomes:

$$\frac{\langle I_{\kappa\lambda} \cdot \Delta_\nu^- f_{\mu\nu}(n) \rangle}{\langle R_{\kappa\lambda} \rangle} = j_\mu^{(\text{static})}(n) \tag{6}$$

where the static current density is

$$j_\mu^{(\text{static})}(n) = e \delta_W \tag{7}$$

We have arrived at what appear to be a discretised version of the continuum Maxwell's equations for the U(1) fields, but satisfied by the expectation values rather than merely in the classical limit of extremising the action. In quantum mechanics, Ehrenfest's theorem relates the time derivative of the position operator to the potential in a way reminiscent of Newton's classical equations of motion, $m \frac{d^2}{dt^2} \langle \hat{\mathbf{x}} \rangle = - \langle \nabla V(\hat{\mathbf{x}}) \rangle$. By analogy, we label the lattice expressions 'Ehrenfest identities'¹.

The theorem defines the lattice field strength operator whose form is dictated by the derivative of the U(1) lattice action. As all actions (in the same universality class) are equivalent in the weak coupling limit, so too are the corresponding definitions of the field strength operator. Much work, however, is performed at finite lattice spacing, and it is

¹A term also used in Zach et. al. [11].

advantageous to avoid extraneous $O(a^2)$ effects by using the ‘correct’ operator. Using this operator we may measure the charge density, which the theorem shows is exactly that of the introduced source. No further charge is induced.

3 Non–Abelian gauge theories

The correct Abelian field strength operator could be calculated as above if the effective Abelian action for the U(1) fields after Abelian projection were known. It is not, and we approach the problem from the full SU(2) action. The SU(2) theory has symmetries analogous to those of the pure Abelian theory, each of which gives rise to Ehrenfest identities. Since these, in the continuum limit, resemble the Euler–Lagrange equations for the corresponding continuum action, we begin by briefly considering these.

The continuum Lagrangian is $\mathcal{L} = \frac{1}{4}G_{\mu\nu}^a(x)G_{\mu\nu}^a(x)$, the isospin index $a \in \{1, 2, 3\}$. Under Abelian projection the third component of the gauge field becomes the Abelian gauge potential. We can rewrite \mathcal{L} to emphasis this:

$$\mathcal{L} = \frac{1}{4} \left(F_{\mu\nu}F_{\mu\nu} + W_{\mu\nu}^*W_{\mu\nu} + \frac{i}{2}F_{\mu\nu} \left(W_\mu W_\nu^* - W_\mu^* W_\nu \right) - \frac{1}{4} \left(W_\mu W_\nu^* - W_\mu^* W_\nu \right)^2 \right) \quad (8)$$

where $F_{\mu\nu}(x) = \partial_\mu A_\nu^3(x) - \partial_\nu A_\mu^3(x)$ is the Abelian field strength, the remaining components forming a complex matter field $W_\mu(x) = A_\mu^1(x) + iA_\mu^2(x)$. This field is electrically charged with respect to the photon; $D_\mu W_\nu(x) = (\partial_\mu - iA_\mu^3(x))W_\nu(x)$, giving $W_{\mu\nu}(x) = D_\mu W_\nu(x) - D_\nu W_\mu(x)$.

Consider the extremisation with respect to $A_\mu^3(x)$. This gives what appear to be Maxwell’s equations with a dynamical, real–valued, conserved electric current formed from the coset (matter) fields and their coupling to the photon.

$$\partial_\nu F_{\mu\nu}(x) = J_\mu^{(\text{dyn})}(x) = \frac{-i}{4} \left(2 \left(W_{\mu\kappa} W_\kappa^* - W_{\mu\kappa}^* W_\kappa \right) - \partial_\kappa \left(W_\kappa W_\mu^* - W_\kappa^* W_\mu \right) \right). \quad (9)$$

The first two terms in the current are precisely what would be expected for a charged vector field, i.e. where $D_\mu W_\mu(x) = 0$. This is not in general true. It is interesting to note, however, that the imposition of this constraint amounts precisely to fixing the theory to the maximally Abelian gauge.

The derivation here relied upon the assumption that we might vary the photon field independently of the charged coset fields. While this is true in the full theory, a gauge fixing constraint couples the variations in the fields. Such an extremisation problem is usually tackled using Lagrange multipliers. The correct lattice operators cannot be predicted by naïve discretisation of continuum results, so we do not pursue this approach here but move on to the lattice Ehrenfest identities.

3.1 Abelian projection on the lattice

After gauge fixing, the $SU(2)$ link matrices may be decomposed in a ‘left coset’ form:

$$U_\mu(n) = \begin{pmatrix} \cos(\phi_\mu(n)) & \sin(\phi_\mu(n))e^{i\gamma_\mu(n)} \\ -\sin(\phi_\mu(n))e^{-i\gamma_\mu(n)} & \cos(\phi_\mu(n)) \end{pmatrix} \begin{pmatrix} e^{i\theta_\mu(n)} & 0 \\ 0 & e^{-i\theta_\mu(n)} \end{pmatrix}, \quad (10)$$

Under a $U(1)$ gauge transformation, $\{g(n) = \exp[i\alpha(n)\sigma_3]\}$,

$$\theta_\mu(n) \rightarrow \theta_\mu(n) + \alpha(n) - \alpha(n + \hat{\mu}) \quad \gamma_\mu(n) \rightarrow \gamma_\mu(n) + 2\alpha(n) \quad (11)$$

In other words, the left coset field derived from the link $U_\mu(n)$ is a doubly charged matter field living on the site n and is invariant under $U(1)$ gauge transformations at neighbouring sites.

The $c_\mu \equiv \cos(\phi_\mu)$ are real-valued fields which near the continuum $\sim 1 + O(a^2)$ where a is the lattice spacing. The off-diagonal $w_\mu \equiv \sin(\phi_\mu)e^{i\gamma_\mu}$ become the charged coset fields $gaW_\mu(x)$, and θ_μ the photon field $gaA_\mu^3(x)$. [The $SU(2)$ coupling $\beta = \frac{4}{g^2}$ in 3+1 dimensions.]

The $SU(2)$ shift symmetries are the left and right multiplications of a link by an arbitrary constant $SU(2)$ matrix, under which the Haar measure is invariant. Since it is the $a = 3$ component that becomes the photon, we consider here only shift matrices of the form $\bar{U} = \exp[i\varepsilon\sigma_3]$.

$$\begin{aligned} \text{right} &: U_\mu(n) \rightarrow U_\mu(n)\bar{U} \equiv \{w_\mu(n) \rightarrow w_\mu(n), \quad \theta_\mu(n) \rightarrow \theta_\mu(n) + \varepsilon\} \\ \text{left} &: U_\mu(n) \rightarrow \bar{U}U_\mu(n) \equiv \{w_\mu(n) \rightarrow w_\mu(n)e^{i2\varepsilon}, \quad \theta_\mu(n) \rightarrow \theta_\mu(n) + \varepsilon\}. \end{aligned} \quad (12)$$

3.2 The identities – no gauge fixing

We first derive the Ehrenfest relations in the simpler, but artificial, context of Abelian projection without gauge fixing, ‘no gauge.’ The $SU(2)$ link matrices combine to form plaquettes $U_{\mu\nu}(n) = U_\mu(n)U_\nu(n + \hat{\mu})U_\mu^\dagger(n + \hat{\nu})U_\nu^\dagger(n)$ and the Wilson action

$$S_W = \sum_{n,\mu<\nu} \left(1 - \frac{1}{2} \text{Tr} U_{\mu\nu}(n)\right). \quad (13)$$

Writing each link as the sum of a diagonal and an off-diagonal matrix, $U = D + O$, we define the product of the diagonal terms around a plaquette

$$Q_{\kappa\lambda}(n') = D_\kappa(n')D_\lambda(n' + \hat{\kappa})D_\kappa^\dagger(n' + \hat{\lambda})D_\lambda^\dagger(n'). \quad (14)$$

and choose as a source term a specific plaquette $P_{\kappa\lambda}(n')$ with real and imaginary parts:

$$\begin{aligned} R_{\kappa\lambda} &\equiv \frac{1}{2} \text{Tr}\{Q_{\kappa\lambda}(n')\} = \left(c_\kappa(n').c_\lambda(n' + \hat{\kappa}).c_\kappa(n' + \hat{\lambda}).c_\lambda(n')\right) \cos \theta_{\kappa\lambda}(n'), \\ I_{\kappa\lambda} &\equiv \frac{1}{2} \text{Tr}\{i\sigma_3 Q_{\kappa\lambda}(n')\} = \left(c_\kappa(n').c_\lambda(n' + \hat{\kappa}).c_\kappa(n' + \hat{\lambda}).c_\lambda(n')\right) \sin \theta_{\kappa\lambda}(n'). \end{aligned} \quad (15)$$

It is for consistency that we include the coset fields. The fields $c_\kappa(n') \rightarrow 1$ in the continuum limit and in the maximally Abelian gauge can be considered to be almost constant [15, 16], since the fluctuations are very small. For this reason, we anticipate only minor changes in the measured averages if the fields $c_\mu(n)$ were excluded from the source as is the case in the traditional Abelian source loop.

The partition function is

$$\mathcal{Z}_{\mathcal{SRC}} = \int [dU_\mu] P_{\kappa\lambda}(n') e^{-\beta S_W} \quad (16)$$

If all fields bar the θ_μ are discarded, the $SU(2)$ action reduces to the $U(1)$ Wilson action, so we define an effective electric charge, e , by $\beta = \frac{1}{e^2}$. The shift invariance gives an analogous result to the $U(1)$ theory

$$j_\mu^{(\text{static})}(n) - \frac{e\beta}{\langle R_{\kappa\lambda} \rangle} \left\langle I_{\kappa\lambda} \frac{\partial S_W}{\partial \varepsilon} \bigg|_{\varepsilon=0} \right\rangle = 0 \quad (17)$$

and the static current density is also based on the localised form of eqn. (4): $j_\mu^{(\text{static})}(n) = e\delta_W$. Although we have introduced an Abelian projected source, the links, U_μ , in the derivative of the action are $SU(2)$ matrices. Were we to replace them also with their

diagonal components only, $D_\mu = \text{diag} (c_\mu e^{i\theta_\mu}, c_\mu e^{-i\theta_\mu})$, we should get a result analogous to the U(1) case:

$$\left. \frac{1}{e} \frac{\partial S_W}{\partial \varepsilon} \right|_{\varepsilon=0; U \rightarrow D} = \frac{1}{e} \Delta_\nu^- I_{\mu\nu}(n) = \Delta_\nu^- f_{\mu\nu}(n) \quad (18)$$

The remaining terms involving both D and O constitute a U(1) gauge invariant dynamical current.

$$\left. \frac{1}{e} \frac{\partial S_W}{\partial \varepsilon} \right|_{\varepsilon=0} = \Delta_\nu^- f_{\mu\nu}(n) - j_\mu^{(\text{dyn})}(n) \quad (19)$$

The terms in $j_\mu^{(\text{dyn})}$ that survive into the continuum limit yield the continuum Euler–Lagrange result (eqn. (9)). The Ehrenfest identity is thus:

$$\frac{\langle I_{\kappa\lambda} \cdot \Delta_\nu^- f_{\mu\nu}(n) \rangle}{\langle R_{\kappa\lambda} \rangle} = j_\mu^{(\text{static})}(n) + \frac{\langle I_{\kappa\lambda} \cdot j_\mu^{(\text{dyn})}(n) \rangle}{\langle R_{\kappa\lambda} \rangle} \quad (20)$$

3.3 Gauge fixing

We restrict the discussion here to the maximally Abelian gauge, defined as the local maximisation by gauge transformations $\{g(n) = \exp[i\alpha_a(n)\sigma_a]\}$ of the Morse functional

$$R[U] = - \sum_{n,\mu} \text{Tr} \left\{ i\sigma_3 \cdot U_\mu(n) \cdot i\sigma_3 \cdot U_\mu^\dagger(n) \right\}. \quad (21)$$

Denoting the gauge transformed link variable as $U_\mu^g(n) \equiv g(n)U_\mu(n)g^\dagger(n + \hat{\mu})$, at a local maximum, assumed to occur at $g = 1$, the first derivative with respect to gauge transformations of $R[U^g]$ will be zero:

$$F^r[U; p] \equiv \left. \frac{\partial R[U^g]}{\partial \alpha_r(p)} \right|_{\alpha=0} = -2\epsilon^{3rc} \text{Tr} \{ i\sigma_c \cdot X(n) \} = 0 \quad (22)$$

where $X(p) = \sum_{\mu > 0} (U_\mu(p) \cdot i\sigma_3 \cdot U_\mu^\dagger(p) + U_\mu^\dagger(p - \hat{\mu}) \cdot i\sigma_3 \cdot U_\mu(p - \hat{\mu}))$. The residual U(1) symmetry is seen in F^3 being trivially zero. The gauge fixed partition function is thus

$$\mathcal{Z}_{SRC}^{gf} = \int [dU_\mu] P_{\kappa\lambda}[U; n'] \cdot e^{-\beta S_W[U]} \cdot \Delta_{FP}[U] \left(\prod_{p,r} \delta(F^r[U; p]) \right) \quad (23)$$

and Δ_{FP} is the Faddeev–Popov gauge fixing operator, a Jacobian to reweight the integration measure after the introduction of the constraint into the partition function:

$$\Delta_{FP}[U] = \left(\int [dg] \prod_{p,r} \delta(F^r[U^g; p]) \right)^{-1} = |\det M_{pq}^{rs}| \quad \text{and} \quad M_{pq}^{rs} \equiv \left. \frac{\partial F^r[U^g; p]}{\partial \alpha_s(q)} \right|_{\alpha=0} \quad (24)$$

is the second term in the Taylor expansion of R [17]. The integral expression is gauge invariant by the invariance of the Haar measure. The determinant is gauge invariant by fiat; we must evaluate it ‘on the constraint’ (i.e. at $F = 0$). If $F \neq 0$ we must first move the configuration along the gauge orbit by gauge transformations until it does.

Applying a shift to a single link and differentiating with respect to the shift parameter, ε , now raises problems, as the shift drives the previously gauge fixed configuration off the constraint. This manifests as derivatives of the constraint δ -functions, which we must avoid. One approach would be to shift not one link, but each of a Polyakov line of links extending around the lattice. By careful choice of the relative sizes of these shifts we may conspire to remove the offending δ -function terms using the constraint equations. This is at the expense of the shift as a local probe of charge density. Instead we accompany the shift of one specific link by an $SU(2)$ gauge transformation over the whole lattice that is also linear in the shift parameter. The combined effect of the shift and ‘corrective’ gauge transformation is to move on a path in configuration space parameterised by ε that remains on the trajectory satisfying the maximally Abelian gauge. There is potentially also a non-locality here, but we shall see that it is very limited in its extent.

The corrective gauge transformation $\{g(q) = 1 + i\varepsilon\eta_s(q)\sigma_s\}$ is calculated using the inverse of the Faddeev–Popov matrix, and so $\eta_3 = 0$:

$$\eta_s(q) = - \sum_{p,r} (M^{-1})_{qp}^{sr} \left. \frac{\partial F^r[U^\varepsilon; p]}{\partial \varepsilon} \right|_{\varepsilon=0} \quad (25)$$

We now apply both the shift and corrective gauge transformation to the partition function in eqn. (23), and differentiate with respect to ε . The action is gauge invariant, and thus only the shift has an effect and the field strength operator and current $j_\mu^{(\text{dyn})}$ are as before (eqn. (19)).

The derivative of the source plaquette now makes two contributions; the shift gives the same static current as before. The second comes from the corrective gauge transformation, and forms a conserved current $j_\mu^{(\text{gauge})}$.

Finally, there are contributions that arise from perturbing the links making up the Faddeev–Popov operator. Under this, the matrix $M \rightarrow M + \varepsilon N$. As we have remarked, the Faddeev–Popov operator is specifically the determinant of M *evaluated on the constraint*. By introducing η we have stayed on the constraint as we shifted the link, and $N = A + B$

has contributions from the shift, A , and from the corrective gauge transformation, B . The derivative forms a further conserved current:

$$j_\mu^{(\text{FP})} = \frac{e}{\Delta_{FP}[U]} \left. \frac{\partial \Delta_{FP}[U^\varepsilon]}{\partial \varepsilon} \right|_{\varepsilon=0} = e \left. \frac{\partial \det(1 + \varepsilon M^{-1}N)}{\partial \varepsilon} \right|_{\varepsilon=0} = e \text{Tr} \{ M^{-1}N \} \quad (26)$$

Defining $j_\mu^{(\text{total})} = j_\mu^{(\text{dyn})} + j_\mu^{(\text{gauge})} + j_\mu^{(\text{FP})}$, we have as the final result a set of identities

$$\frac{\langle I_{\kappa\lambda} \cdot \Delta_\nu^- f_{\mu\nu}(n) \rangle_{gf}}{\langle R_{\kappa\lambda} \rangle_{gf}} = j_\mu^{(\text{static})}(n) + \frac{\langle I_{\kappa\lambda} \cdot j_\mu^{(\text{total})}(n) \rangle_{gf}}{\langle R_{\kappa\lambda} \rangle_{gf}} \quad (27)$$

where the gauge fixed expectation value is defined as

$$\langle \mathcal{O} \rangle_{gf} \equiv \int [dU_\mu] \cdot \mathcal{O} \cdot e^{-\beta S_W[U]} \cdot \Delta_{FP}[U] \left(\prod_{p,r} \delta(F^r[U; p]) \right). \quad (28)$$

We conclude this section with two remarks. In simulation, a lattice average using $\text{SU}(2)$ configurations calculated using standard Monte Carlo methods and then gauge fixed will include the Faddeev–Popov operator in the importance sampling of the measure and it need not be explicitly calculated.

Secondly, when deriving the Faddeev–Popov matrix, there is some ambiguity as to whether one should start from the constraint

$$M_{pq}^{rs} = \left. \frac{\partial F^r[U^g; p]}{\partial \alpha_s(q)} \right|_{\alpha=0} = \left. \frac{\partial}{\partial \alpha_s(q)} \left(\left. \frac{\partial R[U^g]}{\partial \alpha_r(p)} \right|_{\alpha=0} \right) \right|_{\alpha=0} \quad (29)$$

or from the original functional

$$M_{pq}^{rs} = \left. \frac{\partial^2 R[U^g]}{\partial \alpha_s(q) \partial \alpha_r(p)} \right|_{\alpha=0}. \quad (30)$$

Since we believe that M should be a symmetric matrix in general, the latter seems the most natural. In practise, we find that the differences between the two approaches are all multiples of the constraint F and hence when the gauge condition is satisfied the two definitions coincide. Similar ambiguities arise when the response of the Faddeev–Popov matrix to the corrective gauge transformation is considered, since for this we require the third term, L , in the Taylor expansion of R

$$B_{pq}^{rs} = \sum_{u,t} L_{pqu}^{rst} \eta_t(u) \quad (31)$$

and it is unclear whether we should begin from R , F or M in its derivation. Again we find that the most symmetric tensor is derived from R , but that all expressions are the same on the constraint. This we believe to be a general property.

The ambiguities are more serious in the case of the contribution of the shift to the Faddeev–Popov matrix. The matrix A is derived from R by two differentiations with respect to gauge transformations, and one with respect to the shift. The order of these, and at what point variables are set to zero does appear to matter in this case, even when $F = 0$. The correct order of derivatives is as follows:

$$A_{pq}^{rs} = \frac{\partial}{\partial \alpha_s(q)} \left(\frac{\partial}{\partial \varepsilon} \left(\frac{\partial R[U^g]}{\partial \alpha_r(p)} \Big|_{\alpha=0} \right) \Big|_{\varepsilon=0} \right) \Big|_{\alpha=0}. \quad (32)$$

The inner two nested derivatives give the first order shift correction to the constraint, i.e., the second term in $F + \varepsilon F_\varepsilon = 0$. The Faddeev–Popov operator is the matrix of derivatives of this about the constraint and is formed by the outermost nested derivative. The corrected Faddeev–Popov operator must be calculated about the shifted constraint. If, for example, we reverse the order of the outer two nested derivatives we would be calculating the shift of the lowest order Faddeev–Popov operator. This would give an incorrect contribution to the current. Numerical studies confirms these conclusions.

4 Numerical investigation

The terms in the Ehrenfest identities may be measured using Monte Carlo simulation. We begin with a careful test of the expressions, and for this we rewrite eqns. (20,27) to minimise the combination of statistical errors:

$$\begin{aligned} \beta \left\langle I_{\kappa\lambda} \cdot \frac{\partial S_W}{\partial \varepsilon} \Big|_{\varepsilon=0} \right\rangle - k_\mu^{(\text{static})}(n) \cdot \langle R_{\kappa\lambda} \rangle &= 0 \\ \beta \left\langle I_{\kappa\lambda} \cdot \frac{\partial S_W}{\partial \varepsilon} \Big|_{\varepsilon=0} \right\rangle_{gf} - k_\mu^{(\text{static})}(n) \cdot \langle R_{\kappa\lambda} \rangle_{gf} - \left\langle I_{\kappa\lambda} \cdot k_\mu^{(\text{gauge})}(n) \right\rangle_{gf} - \left\langle I_{\kappa\lambda} \cdot k_\mu^{(\text{FP})}(n) \right\rangle_{gf} &= 0 \end{aligned} \quad (33)$$

where the currents k differ from j by a factor of the charge. We further split $k_\mu^{(\text{FP})}$ into the (normally summed) contributions from the shift and corrective gauge transformation. To measure the gauge fixed currents separately requires the inversion of the large Faddeev–Popov matrix. This limits us to the unphysical lattice size of 4^4 for this test, where although the identities must still hold exactly, there may be significant finite volume effects on the

individual currents. For comparison we also test the ‘no gauge’ expression on the same lattice.

In Table 1 we show results where the (left) shifted link is part of the source plaquette (i.e. $|k_\mu^{(\text{static})}| = 1$). The identities are verified within very small statistical errors. The numbers for right shifts are identical within these errors. Off the source, the identities are equally well satisfied.

Table 2 shows the normalised currents arising from shifting a timelike link included in the source, and then one spacelike lattice spacing away from the source in the same plane. In the maximally Abelian gauge we see that on the source the charge of the Abelian projected Wilson loop has been renormalised from unity to a value $O(15\%)$ higher. Further, in the vicinity of the source we find a cloud of like charges has been induced in the vacuum. This is reminiscent of the charge antiscreening (or asymptotic freedom) of the full gauge theory. The charge cloud is localised, falling to near zero by two lattice spacings from the source. We note that in the case of ‘no gauge’ (and also in the Polyakov and diagonal plaquette gauges e.g. ‘ $F_{\mu\nu}$ ’) there is also a renormalisation, but that it reduces the charge of the Abelian Wilson loop. The induced charge cloud is weaker and acts to screen the source, at odds with the behaviour of the full non-Abelian theory. In $U(1)$ we might expect renormalisation and screening, but not in the pure gauge theory.

Assuming the Ehrenfest identities to be true, we may infer $k_\mu^{(\text{total})}$ from a measurement of the derived field strength operator, and break it down into some of its components. This allows the consideration of lattices of a more physically interesting size. In Table 3 we see that the currents are remarkably stable and free of finite volume effects even moving to lattices large enough to support infrared physics. Finite volume effects on the currents induced by a plaquette-sized source would thus appear to be slight.

The Gribov ambiguity was neglected throughout. The Morse functional $R[U]$ typically has a number of local maxima, each giving a different set of maximally Abelian gauge fields corresponding to the Gribov copies. There is some ambiguity regarding which of these to use and one is usually selected randomly during gauge fixing. The variation of gauge variant observables, such as the string tension after Abelian projection, between these copies, although not zero, is small enough that we may continue to neglect the ambiguity in studying the properties of this gauge [18]. It is possible, however, that observables derived

directly from the Faddeev–Popov operator may be unduly sensitive to this ambiguity. Only on lattices comparable in size to the confining length scale (~ 1 fm) do we see nonperturbative effects, and it is on these lattices that $R[U]$ begins to have multiple maxima and Gribov copies appear [18]. The Ehrenfest identities were derived by infinitesimally (in principle) shifting a configuration away from a maximum of $R[U]$, and applying a corrective gauge transformation to return to the maximally Abelian gauge. If the Gribov copies are not too numerous, it seems likely that the configuration does not forsake the attractive influence of the original maximum, and thus returns to the ‘same’ Gribov copy. When the copies become very numerous, however, it may be that two maxima of R are sufficiently proximate that there is a ‘flat direction’ between them, characterised by an extra (near) zero eigenvalue in the Faddeev–Popov matrix.

A 4^4 lattice at $\beta = 2.3$ is far too small to support nonperturbative physics, and indeed the configurations we studied there exhibited only one maximum of R in 1000 gauge fixings (we differentiated the maxima using the value of R and the $U(1)$ plaquette action after Abelian projection, as in [18]). In none of our simulations were there any problems inverting the Faddeev–Popov matrix, and we believe the effects of the Gribov ambiguity to be very small. Only below $\beta \approx 2.15$ do Gribov copies appear, and although we again had no apparent problems inverting, the statistical noise prevents us from making any statements about the currents. On larger lattices, we have shown in Table 3 that the finite volume effects are slight, even on moving to physically large lattices. This does not prove that the Gribov copies have no effect (since we only undertook one gauge fixing per configuration), but the continuing smallness of the statistical errors is perhaps indicative of that.

In a perturbative treatment of the gauge fixed action, the non-local Faddeev–Popov operator would be replaced by a Gaussian integral over propagating ghost fields. In this sense, then, the Faddeev–Popov current that we have isolated may be regarded as the contribution of the ghost fields, hitherto little studied in the maximally Abelian gauge. We conclude that in this context at least these fields play little rôle since the Faddeev–Popov current is only a fraction of the other currents.

5 Summary

In this paper we have exploited symmetries of the lattice partition function to derive a set of exact, non-Abelian identities which define the Abelian field strength operator and a conserved electric current arising from the coset fields traditionally discarded in Abelian projection. The current has contributions from the action, the gauge fixing condition and the Faddeev–Popov operator. Numerical studies on small lattices verified the identity to within errors of a few per cent. We have found the Faddeev–Popov current in particular to be unusually sensitive to systematic effects such as low numerical precision and poor random number generators, but the origin of any remaining, subtle biases, if they exist, is not clear; we have already considered all terms in the partition function.

In a pure U(1) theory the static quark potential may be measured using Wilson loops that correspond to unit charges moving in closed loops, as demonstrated by $|\langle \Delta_\nu^- f_{\nu\mu} \rangle| = \delta_W$. In Abelian projected SU(2) the same measurements in the maximally Abelian gauge yield an asymptotic area law decay and a string tension that is only slightly less than the full non-Abelian value. In other gauges it is not clear that an area law exists — certainly it is more troublesome to identify.

We have seen that in the context of the full theory the Abelian Wilson loop must be reinterpreted. The coset fields renormalise the charge of the loop as measured by $|\langle \Delta_\nu^- f_{\nu\mu} \rangle|$ and charge is also induced in the surrounding vacuum. Full SU(2) has antiscreening/asymptotic freedom of colour charge, and in the maximally Abelian gauge alone have we seen analogous behaviour, in that the source charge is increased and induces charge of like polarity in the neighbouring vacuum. Whether this renormalisation of charge can account for the reduction of the string tension upon Abelian projection in this gauge is not clear. In other gauges, where Abelian dominance of the string tension is not seen, the coset fields appear to have a qualitatively different behaviour, acting to suppress and screen the source charge.

In conclusion, the improved field strength expression defined by the Ehrenfest identity does not coincide with the lattice version of [19] of 't Hooft's proposed field strength operator [20]. The Abelian and monopole dominance of the string tension invites a dual superconductor hypothesis for confinement. If this is to be demonstrated quantitatively such as by verification of a (dual) London equation then a careful understanding of the

field strength operator is required. The Ehrenfest identities may provide this [21].

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ON THE SOURCE	$\beta = 2.3$		$\beta = 2.5$	
	‘no gauge’	MA gauge	‘no gauge’	MA gauge
# meas.	10000	10000	10000	20000
$\beta \left\langle I_{\kappa\lambda} \cdot \frac{\partial S_W}{\partial \varepsilon} \Big _{\varepsilon=0} \right\rangle$	0.0765 (3)	0.6162 (8)	0.0815 (2)	0.68275 (58)
$k^{(\text{static})} \cdot \langle R_{\kappa\lambda} \rangle$	0.0762 (1)	0.5607 (5)	0.0818 (1)	0.63069 (21)
$\langle I_{\kappa\lambda} \cdot k^{(\text{gauge})} \rangle$	—	0.0469 (1)	—	0.04463 (5)
$\langle I_{\kappa\lambda} \cdot k^{(\text{FP})} \rangle$ — shift	—	0.0063 (1)	—	0.00565 (3)
$\langle I_{\kappa\lambda} \cdot k^{(\text{FP})} \rangle$ — corr. g.t.	—	0.0034 (11)	—	0.00133 (51)
LHS eqn. (33)	0.0003 (4)	-0.0011 (17)	-0.0003 (3)	0.00045 (71)

Table 1: Numerical tests of the Ehrenfest identities on 4^4 lattices, with and without gauge fixing.

ON THE SOURCE	$\beta = 2.3$		$\beta = 2.5$	
	‘no gauge’	MA gauge	‘no gauge’	MA gauge
$k^{(\text{dyn})}$	-0.1266 (40)	0.0800 (5)	-0.1293 (25)	0.0781 (7)
$k^{(\text{gauge})}$	—	0.0836 (2)	—	0.0711 (1)
$k^{(\text{FP})}$	—	0.0173 (20)	—	0.0112 (3)
$k^{(\text{static})} + k^{(\text{total})}$	0.8734 (40)	1.1809 (20)	0.8707 (25)	1.1604 (10)

OFF THE SOURCE	$\beta = 2.3$		$\beta = 2.5$	
	‘no gauge’	MA gauge	‘no gauge’	MA gauge
$k^{(\text{dyn})}$	-0.0074 (40)	0.0139 (5)	-0.0126 (40)	0.0145 (5)
$k^{(\text{gauge})}$	—	0.0042 (1)	—	0.0037 (1)
$k^{(\text{FP})}$	—	0.0001 (25)	—	0.0012 (3)
$k^{(\text{static})} + k^{(\text{total})}$	-0.0074 (40)	0.0182 (25)	-0.0126 (40)	0.0194 (10)

Table 2: Ehrenfest currents $\langle I_{\kappa\lambda} \cdot k \rangle / \langle R_{\kappa\lambda} \rangle$ on 4^4 lattices, with and without gauge fixing.

ON THE SOURCE	$j^{(\text{total})}$	$j^{(\text{dyn})}$	$j^{(\text{gauge})} + j^{(\text{FP})}$
$\beta = 2.3, L = 4$	0.1809 (20)	0.0800 (5)	0.1009 (25)
$\beta = 2.3, L = 6$	0.1776 (10)	0.0674 (5)	0.1102 (5)
$\beta = 2.3, L = 8$	0.1735 (39)	0.0642 (25)	0.1093 (41)
$\beta = 2.3, L = 10$	0.1840 (83)	0.0654 (30)	0.1186 (94)
$\beta = 2.5, L = 4$	0.1596 (7)	0.0781 (7)	0.0824 (13)
$\beta = 2.5, L = 6$	0.1610 (15)	0.0757 (17)	0.0853 (17)
$\beta = 2.5, L = 10$	0.1658 (40)	0.0753 (16)	0.0905 (35)

Table 3: Finite volume effects on the Ehrenfest currents, $\langle I_{\kappa\lambda}.k \rangle_{gf} / \langle R_{\kappa\lambda} \rangle_{gf}$ in the maximally Abelian gauge.